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Friction Factors and Pressure Drop for Sinusoidal Laminar Flow of Water and Blood in Rigid Tubes

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From the Navier-Stokes equations and a modified Fanning equation, a theoretical equation was derived for computing friction factors and pressure drop for sinusoidal flow in rigid pipes. The friction factor equation was $f = (\pi/16S)(16/N_{Re})$, which is analogous to the usual laminar flow equation. The factor S is dependent on the frequency and kinematic viscosity and is easily computed. Friction factors were calculated from experimental data and it was found that the theoretical friction factors predicted the experimental values to within less than 5%.

Pulsatile flow has received an increased amount of attention from engineers and physiologists in recent years, who recognize that the pulsatile flow phenomenon exists in pumping systems, heat and mass transfer operations as well as in the circulatory blood flow circuit in living organisms.

Chantry et al. (*3*) applied pulsation to liquid-liquid extraction and De Maria and Benenati (*5*) studied the effect of pulsation in batch thermal diffusion. Krasuk and Smith (*9*) and Shirotsuka (*17, 18*) studied mass transfer in a pulsed column and Linford (*13*) verified that the superimposed pulsation did not affect the streamline character of flow. Many investigators reported that the pulsation increased the efficiency of the mass and heat transfer processes.

Theoretical studies of pulsatile flow in a circular tube have been done by Sexl (*16*), Uchida (*22*), Womersley (*25*), Lambossy (*11*), and Kusama (*10*). Womersley and others derived the equation for average flow rate starting with the Navier-Stokes equation and a sinusoidal pressure gradient $dp/dz = A\, e^{i\omega t}$.

Physiologists have been interested in pulsatile flow because of its applicability to the circulatory system. Landowne (*12*) and Taylor (*21*) studied the propagation of a pulse wave in arteries and Bergel (*1*) investigated the dynamic elastic properties of the wall during pulsatile flow. McDonald (*15*), Evans (*6*), and Caro (*2*) established a relationship between pressure and flow in arteries by using Womersley's theory (*23, 24*).

To determine the power requirement for designing a pulsatile flow system, it is important to find the energy loss expressed as the friction at the tube wall. Shirotsuka (*17, 18*) studied mass, heat, and momentum transfer in pulsatile flow. He proposed to represent the fractional increments of pulsatile friction factor vs. steady flow values by the nondimensional empirical equation

$$\frac{f_p - f}{f} = 2.5 \times 10^3 \left(\frac{A\,n}{u}\right)^{1.6} \left(\frac{A\,n\,D}{\alpha}\right)^{-0.6} \left(\frac{D}{A\,n}\right)^{0.5}$$

Kusama (*10*) determined the time-averaged friction factor f_p for pulsatile flow by finding the work necessary to overcome the friction force over the period T of a pulse.

All of the preceding derivations involved extensive calculations and also required steady flow values to determine the friction factor for pulsatile flow. The work of Hershey and Song (*20*) was undertaken to develop a theoretical equation for pulsatile flow which would be analogous to the steady flow laminar friction factor equation

$$f = 16/N_{Re}$$

DERIVATION OF A FRICTION FACTOR EQUATION FOR SINUSOIDAL LAMINAR FLOW

If the upstream pressure is $P_0\,(1 + \sin \omega t)$, then for an incompressible fluid in a rigid tube, the downstream pressure may be expressed by a modified Fanning equation:

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$$P = \left(P_0 - \frac{f_p \rho <u>^2 z}{R g_c}\right)(1 + \sin \omega t) \quad (1)$$

Equation (1) can be seen to reduce to the Fanning equation when the frequency is zero. The Navier-Stokes equations in cylindrical coordinates for the z component of laminar flow can be shown to reduce to Equation (2):

$$\rho \frac{\partial u}{\partial t} = -g_c \frac{\partial P}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) \quad (2)$$

If a new time scale is defined, $\omega t = \omega \tau + 3\pi/2$, then Equation (1) is transformed to

$$P = \left(P_0 - \frac{f_p \rho <u>^2 z}{R g_c}\right)(1 - \cos \omega\tau) \quad (3)$$

where the *initial* condition in Equation (2) is now

$$u(r, \tau)\big|_{\tau=0} = 0$$

since

$$\Delta P = \left(\frac{f_p \rho <u>^2 z}{R g_c}\right)(1 - \cos \omega\tau)\bigg|_{\tau=0} = 0$$

If now Equation (3) is differentiated with respect to z and substituted into Equation (2), the result is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{\alpha}\frac{\partial u}{\partial \tau} = -k(1 - \cos \omega\tau) \quad (4)$$

where

$$\alpha = \mu/\rho \quad (5)$$

and

$$k = \frac{f_p <u>^2}{R\alpha} \quad (6)$$

If the Laplace transform of Equation (4) is taken with respect to τ (*14, 19*), Equation (4) becomes

$$\frac{d^2\bar{u}}{dr^2} + \frac{1}{r}\frac{d\bar{u}}{dr} - \frac{S}{\alpha}\bar{u} = -k\left(\frac{1}{S} - \frac{S}{\omega^2 + S^2}\right) \quad (7)$$

The solution to this inhomogeneous zero-order Bessel differential equation is given (*3*) as

$$\bar{u}(r, S) = C_1 J_0\left(i\sqrt{\frac{S}{\alpha}}\,r\right) + C_2 Y_0\left(i\sqrt{\frac{S}{\alpha}}\,r\right) + \frac{\alpha k}{S}\left(\frac{1}{S} - \frac{S}{\omega^2 + S^2}\right) \quad (8)$$

with boundary conditions

$$u(0, \tau) = \text{finite} \rightarrow \bar{u}(0, s) = \text{finite}$$

$$u(R, \tau) = 0 \rightarrow \bar{u}(R, s) = 0$$

The constants C_1 and C_2 can be evaluated and the result is Equation (9):

$$\bar{u}(r, S) = \frac{\alpha k}{S}\left(\frac{1}{S} - \frac{S}{\omega^2 + S^2}\right)\left[1 - \frac{J_0\left(i\sqrt{\frac{S}{\alpha}}\,r\right)}{J_0\left(i\sqrt{\frac{S}{\alpha}}\,R\right)}\right] \quad (9)$$

The inverse Laplace transform of Equation (9) yields

$$u(r, \tau) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \alpha k\left(\frac{1}{S} - \frac{S}{\omega^2 + S^2}\right) \frac{J_0\left(i\sqrt{\frac{S}{\alpha}}\,R\right) - J_0\left(i\sqrt{\frac{S}{\alpha}}\,r\right)}{S\, J_0\left(i\sqrt{\frac{S}{\alpha}}\,R\right)} \exp\{S\tau\}\, dS \quad (10)$$

By defining

$$f(s) = \alpha k\left(\frac{1}{S} - \frac{S}{\omega^2 + S^2}\right) \quad (11)$$

$$g(s) = \frac{J_0\left(i\sqrt{\frac{S}{\alpha}}\,R\right) - J_0\left(i\sqrt{\frac{S}{\alpha}}\,r\right)}{S\, J_0\left(i\sqrt{\frac{S}{\alpha}}\,R\right)} \exp\{S\tau\} \quad (12)$$

Equation (10) can be solved for the inverse Laplace transform by the convolution integral

$$u(r, \tau) = \int_0^\tau f(\tau')\, g(\tau - \tau')\, d\tau' \quad (13)$$

From standard mathematical tables (*8*), Equation (11) can be transformed to Equation (14):

$$f(\tau) = k(1 - \cos \omega\tau) \quad (14)$$

From the residue theorem (*4*), the poles of $g(s)$ in Equation (12) are

$$J_0\left(i\sqrt{\frac{S}{\alpha}}\,R\right) = 0$$

and

$$g(\tau) = \sum_{n=1}^{\infty} \frac{\left[J_0\left(i\sqrt{\frac{S}{\alpha}}\,R\right) - J_0\left(i\sqrt{\frac{S}{\alpha}}\,r\right)\right]\exp\{S\tau\}}{J_0\left(i\sqrt{\frac{S}{\alpha}}\,R\right) - \frac{1}{2}\left(i\sqrt{\frac{S}{\alpha}}\,R\right) J_1\left(i\sqrt{\frac{S}{\alpha}}\,R\right)}\Bigg|_{S = -\frac{\alpha\mu_n^2}{R^2}} \quad (15)$$

or

$$g(\tau) = 2\sum_{n=1}^{\infty} \frac{J_0\left(\frac{\mu_n}{R}r\right)}{\mu_n J_1(\mu_n)} \exp\left\{\frac{-\alpha\mu_n^2}{R^2}\tau\right\} \quad (16)$$

where $\mu_n = i\sqrt{\frac{S}{\alpha}}\,R$ and $J_0(\mu_n) = 0$. Therefore from Equations (13), (14), and (16)

$$u(r, \tau) = 2\alpha k \sum_{n=1}^{\infty} \frac{J_0\left(\frac{\mu_n}{R}r\right)}{\mu_n J_1(\mu_n)}\left[\frac{R^2}{\alpha\mu_n^2} - \frac{\alpha R^2 \mu_n^2 \cos \omega\tau + R^4 \omega \sin \omega\tau}{\omega^2 R^4 + \alpha^2 \mu_n^4} - \left(\frac{R^2}{\alpha\mu_n^2} - \frac{\alpha \mu_n^2 R^2}{\omega^2 R^4 + \alpha^2\mu_n^4}\right)\right] \exp\left\{\frac{-\alpha\mu_n^2}{R^2}\tau\right\} \quad (17)$$

Equation (17) can be shown (*20*) to reduce to the equivalent steady flow equation when the frequency goes to zero. From Equation (17) and the mean value theorem, the average velocity can be derived:

$$<u> = \frac{\int_0^{2\pi}\int_0^{2\pi/\omega}\int_0^R u\, r\, dr\, d\tau\, d\theta}{\int_0^{2\pi}\int_0^{2\pi/\omega}\int_0^R r\, dr\, d\tau\, d\theta} \quad (18)$$

or

$$<u> = \frac{2k\,R^2}{\pi} \sum_{n=1}^{\infty} \left[\frac{2\pi}{\mu_n{}^4} - R^2\omega \left(\frac{1}{\alpha\mu_n{}^6} - \frac{\alpha}{\omega^2\,R^4\,\mu_n{}^2 + \alpha^2\,\mu_n{}^6} \right) \left(1 - \exp\left\{ \frac{-\,2\pi\alpha\mu_n{}^2}{R^2} \right\} \right) \right] \quad (19)$$

Equation (19) can also be shown (*20*) to reduce to the Hagen-Poiseuille equation when the frequency is zero. Substitution of Equation (6) into Equation (19) and with rearrangement, it can be shown (*20*) that the friction factor is expressed by Equation (20):

$$f_p = \frac{\pi^2\,\alpha\,R}{2Q}\;\frac{1}{\displaystyle\sum_{n=1}^{\infty}\left[\frac{2\pi}{\mu_n{}^4} + \frac{R^2\,\alpha\,\omega}{\omega^2\,R^4\,\mu_n{}^2 + \alpha^2\,\mu_n{}^6} - \frac{R^2\,\omega}{\alpha\mu_n{}^6}\right]} \quad (20)$$

If a dimensionless parameter λ is defined as

$$\lambda = \frac{R^2\,\omega}{\alpha} \quad (21)$$

along with

$$S = \sum_{n=1}^{\infty} \left[\frac{2\pi}{\mu_n{}^4} + \frac{1}{\lambda\mu_n{}^2 + \dfrac{\mu_n{}^6}{\lambda}} - \frac{\lambda}{\mu_n{}^6} \right] \quad (22)$$

and

$$N_{Re} = \frac{2R\,<u>}{\alpha} \quad (23)$$

Equation (20) becomes

$$f_p = \frac{\pi}{S}\,\frac{1}{N_{Re}} = \left(\frac{\pi}{16S}\right)\left(\frac{16}{N_{Re}}\right) \quad (24)$$

Thus the friction factor for sinusoidal pulsatile flow f_p in Equation (24) can be calculated from the average Reynolds number and the factor $(\pi/16S)$, which is a function of the frequency factor λ. It can be shown rigorously that $(\pi/16S) = 1$ when the frequency goes to zero. It is now an easy matter to find f_p, since S converges very rapidly. Figure 1 is a plot of the pulsatile friction factor as a function of Reynolds number with λ as the parameter ($\lambda = 0$ for steady flow).

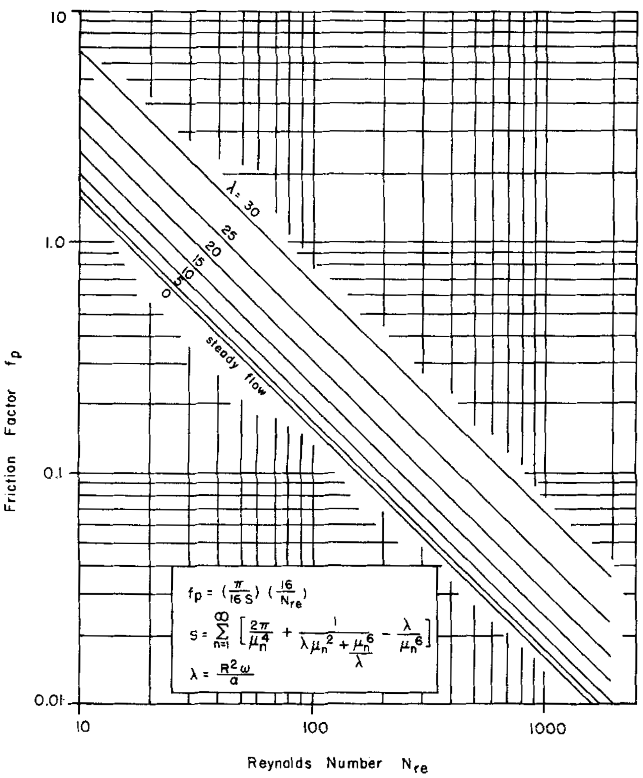

Fig. 1. Friction factor vs. Reynolds number predicted theoretically.

DERIVATION OF A GENERALIZED FRICTION FACTOR EQUATION FOR ANY PRESSURE PULSE $F(t)$

If now the pulsatile pressure component is not sinusoidal, but any function $F(t)$, then by analogy with Equation (1)

$$P = \left(P_0 - \frac{f_p\,\rho <u>^2\,z}{R\,g_c}\right) F(t) \quad (25)$$

By mathematical operations similar to the previous equation, it can be shown (*20*) that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r} - \frac{1}{\alpha}\,\frac{\partial u}{\partial t} = -\frac{f_p\,\rho <u>^2}{\mu\,R}\,F(t) = -\,k\,F(t) \quad (26)$$

with initial and boundary conditions

$$u(r,\ 0) = 0 \quad (26)$$

$$u(0, t) = \text{finite} \quad (27)$$

$$u(R, t) = 0 \quad (28)$$

It can be shown (*20*) that the velocity profile $u(r, t)$ is given by Equation (29):

$$u(r, t) = \sum_{n=1}^{\infty} \frac{2\,\alpha k\,J_0\left(\dfrac{\mu_n}{R}\,r\right)}{\mu_n\,J_1\,(\mu_n)} \exp\left\{\frac{-\,\alpha\,\mu_n{}^2}{R^2}\,t\right\} \int_0^t F(t')\,\exp\left\{\frac{\alpha\,\mu_n{}^2}{R^2}\,t'\right\} dt' \quad (29)$$

where once again μ_n's are the roots of $J_0(\mu_n) = 0$.

By continuing the analogy with the previous equations for sinusoidal pulsatile flow, the friction factor becomes

$$f_p = \frac{\pi R^4 t_c}{4\mu Q}\;\frac{1}{\displaystyle\sum_{n=1}^{\infty}\left[\frac{1}{\mu_n{}^2}\int_0^{t_c}\int_0^{t} F(t')\,\exp\left\{\frac{-\,\alpha\,\mu_n{}^2(t - t')}{R^2}\right\} dt'dt\right]} \quad (30)$$

where

$$Q = \pi R^2 <u>$$

EXPERIMENT

It was previously shown that the friction factor for sinusoidal laminar flow of a Newtonian fluid could be theoretically calculated from

$$f_p = \left(\frac{\pi}{16S}\right)\left(\frac{16}{N_{Re}}\right) \quad (24)$$

where $(\pi/16S)$ is a frequency dependent correction factor which can be calculated without experimental data. It was shown that S was a function of $\lambda = R^2w/\alpha$ and hence f_p was dependent upon λ values. The theoretical f_p values can therefore be compared with experimental friction factors, calculated from Equation (*31*), a time-averaged Fanning equation

$$f_p = \frac{(\Delta p)_{\text{avg}}\,R\,g_c}{z\,\rho <u>^2} \quad (31)$$

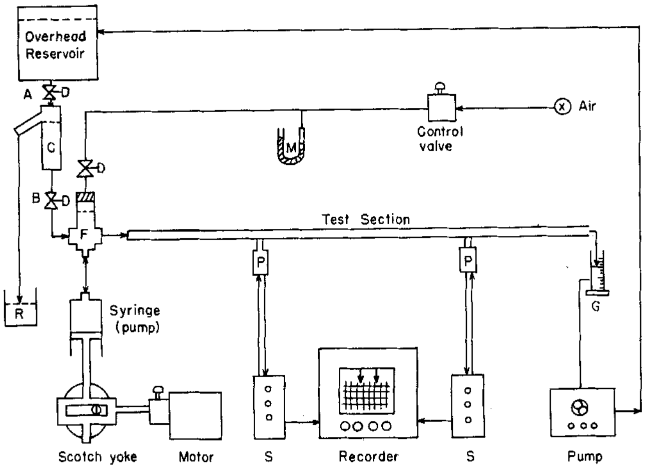

Fig. 2. Experimental flow diagram.

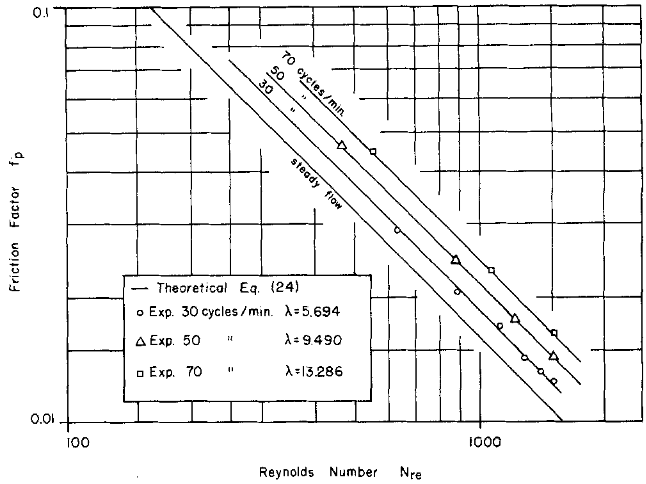

Fig. 3. Friction factors for water in tube 3 at 25°C.

where

$$(\Delta P)_{avg} = \frac{\int_o^{2\pi/\omega} (\Delta P)\, dt}{\int_o^{2\pi/\omega} dt} \quad (32)$$

and (ΔP) is the experimental pressure drop at time t in the cycle having a periodic time $2\pi/\omega$. Thus the experimental procedure was designed to collect sinusoidal pressure and flow data and to calculate and compare friction factors from Equations (24) and (31) and also from the corresponding steady flow average friction factor $f = 16/N_{Re}$.

The experimental flow diagram is given in Figure 2. The test section consisted of Pyrex rigid tubes with diameters ranging from 0.2 to 0.8 cm. These tubes were about 1 meter long with two holes drilled 75 cm. apart for pressure taps. Glass tubes 1 in. long and 0.3 cm. wide were cemented into the pressure taps. To these 1-in. glass tubes were connected pressure transducers (Statham type PG 132 TC), shown in Figure 2 as item *P*. For the steady flow component, there was an overhead constant head reservoir *C* (0.5 gal.). The overflow from column *C* and the exiting fluid from the test section were recycled. To produce a sinusoidal pulsatile flow by superposition of a pulsatile component on a steady flow, a pulse generator was connected to the steady flow line as shown in Figure 2. The pulse was generated by a syringe (20 cc.) which was used as a valveless pump. A ⅛ h.p. variable speed motor (General Electric, 0 to 400 rev./min.) was connected to a scotch yoke which was connected to the syringe. To diminish the distortion of the pressure pulse, and also to function as a four-way connector, an air pocket *F* was inserted before the test section.

The instantaneous pressure at both taps was measured by pressure transducers *P* which were excited by power supplies, S (Primary Precision Measurements Division, model SG 8-3) and recorded on a dual channel recorder (Varian, model G-22A). The average flow rate was measured directly with a graduated cylinder and stop watch. To calibrate the pressure transducers frequently, a compressed air line was connected by valve *D* to *F* and a mercury manometer *M*. Nalgon tubing, Voplex Product P 73-223 8000 B4 and L1 were used for all connections. Detailed technical data are given elsewhere (*20*).

Seven different frequencies, 10 to 70 cycles/min., were used with amplitudes from 1 to 11 cc./cycle. The maximum average flow corresponded to $N_{Re} = 1{,}300$ in every tube. The pressure pulse frequency was varied by adjusting the screw-type motor speed controller. The amplitudes were changed by adjusting the scotch yoke stroke. After adjustment of the motor speed and the scotch yoke at desired values, a steady flow was started simply by opening valves *A* and *B*. A pressure pulse was superimposed on this steady flow by starting the motor which drives the scotch yoke. Valve *B* was adjusted until the pressure pulse curves showed oscillations above a minimum value of zero. The corresponding steady flow was obtained by stopping the pulse generator. The dimensions of the tubes are given in Table 1. The fluid was water for one phase of the work and blood for the other.

TABLE 1. DIMENSIONS OF PYREX GLASS TUBE

Tube No.	Cross-sectional area, sq. cm.	Radius *R*, cm.	Length between taps *L*, cm.	Total length of tube, cm.
2	0.03030	0.098213	75.0	105.0
3	0.05102	0.12744	75.1	101.00
4	0.09662	0.17537	79.2	110.50
5	0.1449	0.21478	75.0	120.00
6	0.31148	0.31487	75.0	118.50
8	0.500	0.39894	79.2	119.00

DISCUSSION OF RESULTS

It was assumed throughout this work that the pulsatile flow was laminar. Linford (*13*) and Krasuk and Smith (*9*) reported that for low Reynolds numbers and low frequencies, the superimposed pulsation did not affect the streamline character of the steady flow. To study this drops of red oil with a density approximately the same as the liquid were introduced with a hypodermic needle. It was observed that the drops of oil appeared to move in an axial direction with no apparent radial displacement.

The calculated and experimental friction factor values were plotted as a log-log graph from Equation (24)

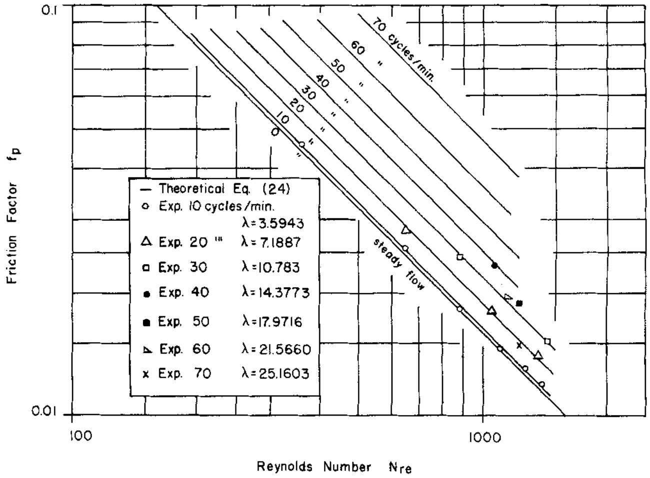

Fig. 4. Friction factors for water in tube 4 at 25°C.

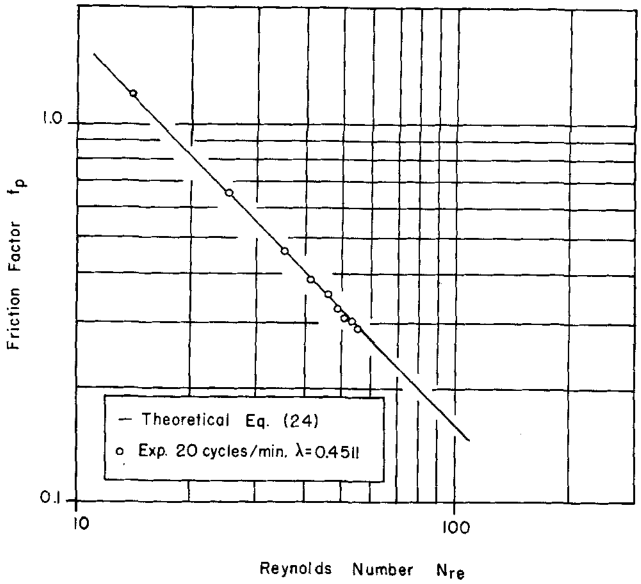

Fig. 5. Friction factors for blood in tube 2 at 25°C.

(theoretical), $f = 16/N_{Re}$ (steady flow), and Equation (31) (experimental). Some representative results are given in Figures 3, 4, 5, and 6.

The friction factor predicted from Equation (24) represented the experimental results quite well; the difference was less than 5%. For high frequencies (λ values greater than 17), the experimental friction factor was lower than the theoretical values calculated from Equation (24). Shirotsuka (*18*) attempted to explain these smaller than expected friction factors at high frequencies by inferring distortion in the boundary layer. It may also be that the inertial effects upon flow become more pronounced than expected as the frequency increases (λ increases). It was also found that the pulsatile friction factor approached the steady flow friction factor, $f = 16/N_{Re}$ with decreasing tube size ($\lambda \to 0$). For these small tubes at low values of λ, different frequencies had a very small effect on the pulsatile flow friction factor. In summary, based on all the data it can be concluded that friction factors predicted by Equation (24) are generally more accurate than the $f = 16/N_{Re}$ approach and actually predict the experimental results satisfactorily within 5%. The critical parameter is λ. In comparing the predicted results with $f = 16/N_{Re}$ it should be pointed out that $f = 16/N_{Re}$ is only applicable strictly to steady flow conditions. The sinusoidally varying pressure imposed here results in accelerating and decelerating flows for which the Hagen-Poiseuille relationship is only approximately followed.

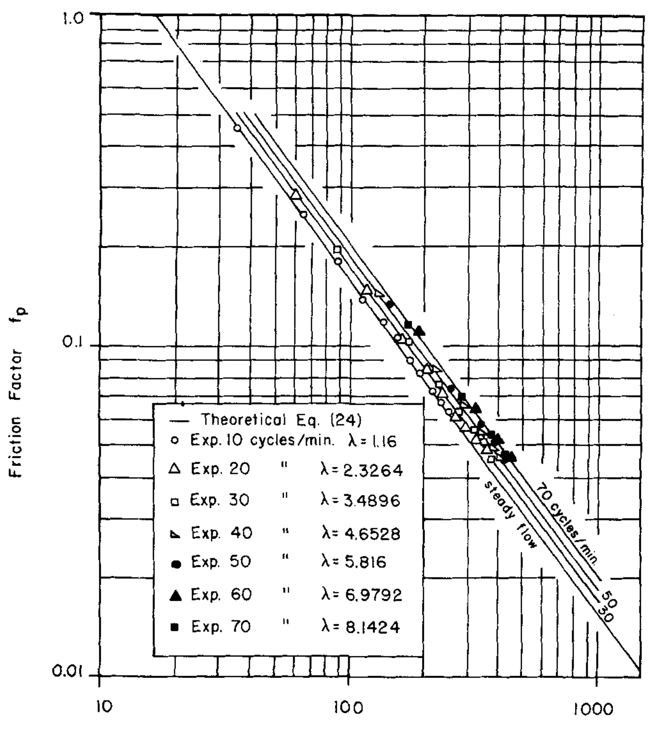

Fig. 6. Friction factors for blood in tube 5 at 25°C.

PREDICTION OF AVERAGE PRESSURE DROP FOR SINUSOIDAL LAMINAR FLOW IN RIGID TUBES

Previously it was shown that the pulsatile friction factor could be predicted within 5% for water and blood flowing in rigid tubes. Since it is usually of a great interest to determine the pressure loss for steady fluid flow, as well as for pulsatile flow, it is desirable to predict the pressure drop.

It has been shown that the friction factor for sinusoidal flow is given by

$$f_p = \left(\frac{\pi}{16\,S}\right)\left(\frac{16}{N_{Re}}\right) \tag{24}$$

based on

$$\Delta P = \left(\frac{f_p\,\rho <u>^2\,z}{R\,g_c}\right)(1 + \sin \omega t) \tag{33}$$

From Equations (24) and (33) it can be shown (*20*) that the average pressure drop can be predicted from Equation (34) without resort to experiment.

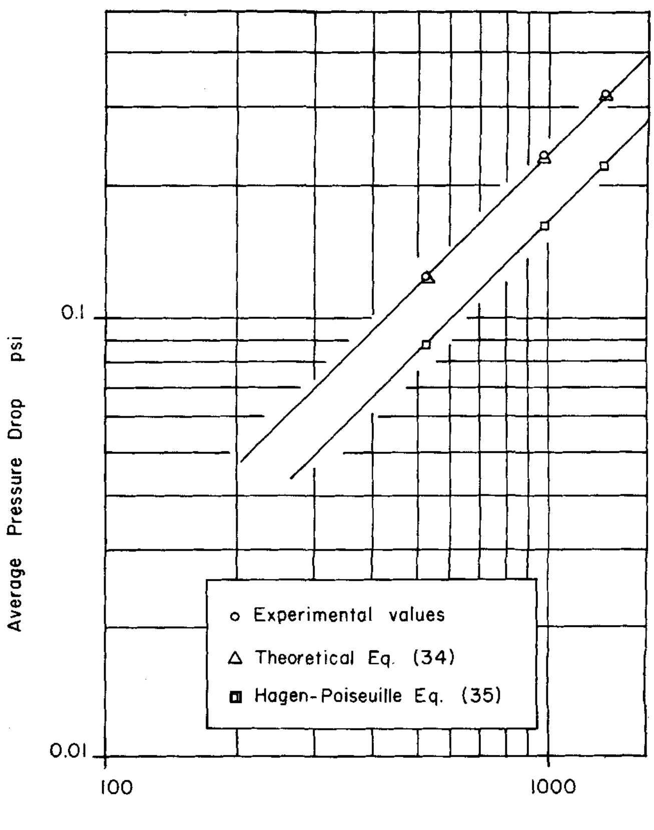

Fig. 7. Pressure drop for water in tube 3 at 25°C.

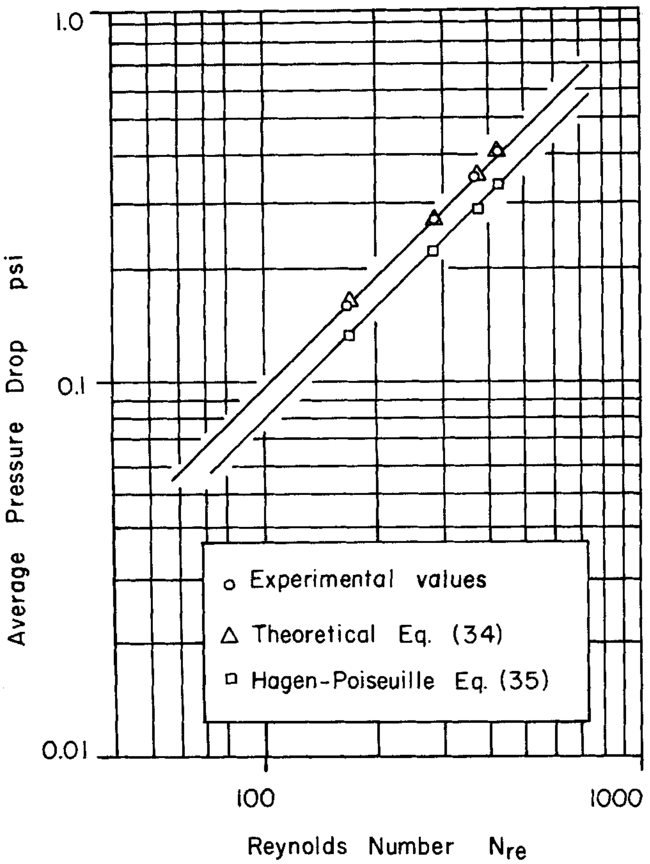

Fig. 8. Pressure drop for blood in tube 5 at 25°C.

$$(\Delta P)_{\text{avg}} = \frac{\int_0^{2\pi/\omega} (\Delta P) dt}{\int_0^{2\pi/\omega} dt} = \frac{f_p \, \rho <u>^2 z}{R \, g_c} \quad (34)$$

These predictions can then be compared with experimentally measured average pressure drop. These two $(\Delta P)_{\text{avg}}$ relations can also be compared with the "averaged" flow results, the Hagen-Poiseuille prediction of pressure drop with $f = 16/N_{Re}$

$$(\Delta P)_{\text{avg}} = \frac{8 \, \mu \, z <u> \rho}{R^2 \, g_c} \quad (35)$$

PRESENTATION OF RESULTS

The pulsatile pressure drop data are shown in Figures 7 and 8. It was found that the differences between the theoretical $(\Delta P)_{\text{avg}}$ from Equation (34) and actual experimental values were less than 6% for lower frequencies and less than 3% for frequencies higher than 30 cycles/min. The differences between experimental values and the "steady flow" $f = 16/N_{Re}$ approach of Equation (35) were less than 5% for lower frequencies and more than 11% for higher frequencies of 40 cycles/min.

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NOTATION

A = linear amplitude of pulsation
C_1, C_2 = constants
D = diameter of the tube
f, f_p = friction factors for steady and pulsatile flows, respectively
$F(t)$ = pressure function as defined by Equation (25)
$f(t)$ = function of time as defined by Equation (4)
g_c = 32.174($(\text{lb.}_f)(\text{ft.})/(\text{lb.}_m)(\text{sec.}^2)$
$g(t)$ = function of time as defined by Equation (15)
J_0 = Bessel function of the first kind of order zero
k = defined by Equation (6)
N_{Re} = Reynolds number
n = frequency
P, P_0 = pressure
ΔP = pressure gradient
$(\Delta P)_{\text{avg}}$ = average pressure drop for pulsatile flow
Q = average flow rate
R = radius of the tube
r = axis of radial direction
S = sum of infinite series as defined by Equation (22)
t, t' = time
t_c = periodic time interval for measuring flow
u = fluid velocity
$<u>$ = average velocity
$\overline{u}$ = transformed velocity
Y_0 = Bessel function of the second kind of order zero
z = axial tube distance

Greek Letters

α = kinematic viscosity, μ/ρ
θ = angular direction coordinate
λ = dimensionless parameter equal to $R^2\omega/\alpha$
μ = fluid viscosity
μ_n = roots of zero-order Bessel function
ρ = fluid density
ω = angular velocity, $\omega = 2\pi n$
τ = time defined by $\omega t = \omega\tau + (3\pi/2)$

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